*International Journal of Theoretical Physics, Vol. 27, No. 1, 1988*

# Ratios of Magnetic Charge to Charge and of Magnetic Mass to Mass

**Anne M. R. Magnon**[1]

*Received August 6, 1987*

It is shown that, unlike the case of (vacuum) solutions describing isolated bodies, conformal Killing fields are not excluded by the structure of vacuum gravitational magnetic monopoles at null infinity. The resulting dilation must be constant. This brings support to the viewpoint that such solutions might have a role to play in the understanding of gravitational entropy and time's arrow. If, in addition, a Maxwellian magnetic monopole (Dirac string singularity) is available, the ratio of the total magnetic charge (magnetic mass) over the total electric charge (mass) can be identified. This common feature between the gravitational and the electromagnetic interaction finds its origin in the space-time topology.

## 1. INTRODUCTION

Although much attention has been devoted to isometries admitted by solutions (to Einstein's solution) describing isolated gravitating bodies, i.e., asymptotically flat space-times, little has been done on more exotic solutions emerging from the availability of specific nontrivial topologies, such as those of the NUT or KERR-NUT family (supposed to describe the gravitational analogues of the (Dirac) Maxwellian magnetic monopoles).

We focus here on such solutions and show that conformal isometries are not excluded by their structure (Magnon, 1986a–c; Magnon and Guo, 1987) at null infinity (compact $\mathscr{I}$), which is not the case for isolated bodies with an $S^2\times R$ topology at $\mathscr{I}$). However, the dilation parameter must be constant. As a result of our analysis, we are able to set a relation between the ratio of the total magnetic mass and total mass of such a space-time and its Maxwellian analogue: a specific source-free Maxwell field can be selected via the structure (characteristic data) at null infinity, which provides the identification of these ratios. This brings out a common feature between the gravitational and electromagnetic interactions (Magnon, 1986a–c,

[1]Département de Mathématiques, Université de Clermont-Ferrand, 63170 Aubière, France.

1987a,b; Magnon and Guo, 1987). Since gravitational magnetic monopoles could play a role in the definition of gravitational entropy or time's arrow (Magnon, 1986a–c, 1987a; Magnon and Guo, 1987), the availability of a conformal isometry inducing constant dilation could enable further investigations.

## 2. PRELIMINARIES AND NOTATION

Let $(\tilde{M}, \tilde{g}_{ab})$ be a space-time, i.e., a four-dimensional manifold with smooth metric $\tilde{g}_{ab}$.

We assume the existence of a conformal Killing vector field $\xi^a$:

$$\mathscr{L}_\xi \tilde{g}_{ab} = k\tilde{g}_{ab} \tag{1}$$

for some scalar field $k$ on $\tilde{M}$. If $k=0$, $\xi^a$ is a Killing vector field.

We shall be concerned here with space-times that are asymptotically NUT at conformal null infinity. Such space-times are suitable for the description of gravitational magnetic monopoles, have been investigated in Magnon (1987a), and can be characterized by a manifold $M$ with boundary $\mathscr{I}$, smooth metric $g_{ab}$, and smooth scalar field $\Omega$ (on $M$), and a mapping from $\tilde{M}$ to $M$ such that

(i) On $\tilde{M}$, $g_{ab}=\Omega^2\tilde{g}_{ab}$.
(ii) On $\mathscr{I}$, $\Omega=0$, $\nabla_a\Omega\neq 0$ $g^{ab}\,\nabla_a\Omega\,\nabla_b\Omega=0$.

$(\tilde{M}, \tilde{g}_{ab})$ is called the physical space-time, $(M, g_{ab}, \Omega)$ an asymptote of $(\tilde{M}, \tilde{g}_{ab})$, and $\mathscr{I}$, null infinity, is a boundary added to the physical space-time at infinite distance in null directions. The topology of $\mathscr{I}$ is that of a $U(1)$ fiber bundle over $\mathscr{S}$, the 2-sphere of its null generators (Hopf fibration over $S^2$) due to nonvanishing of the second cohomology class of the space-time manifold $\tilde{M}$.

Various fields can be defined or pulled back on $\mathscr{I}$. Let $i^*$ denote the pullback operation. Define $n_a=\nabla_a\Omega$. One has the following fields on $\mathscr{I}$.

1. $\underline{g}_{ab}=i^*(g_{ab})$, a degenerate metric with signature $(0,+,+)$.
2. $\underline{n}^a=i^*(n^a)$, the null generators of $\mathscr{I}$.
3. $\underline{\varepsilon}^{abc}\equiv i^*(\varepsilon^{abcd}n_d)$, which can be reduced to the condition $\underline{\varepsilon}^{abc}\underline{\varepsilon}_{abc}=6$.
4. $\underline{K}_{abcd}=i^*(\Omega^{-1}C_{abcd})$, the spin 2 field on $\mathscr{I}$ that encompasses the falloff of the Weyl tensor $C_{abcd}$ along the null directions.
5. $\underline{K}^{ab}=\underline{\varepsilon}^{amn}\underline{\varepsilon}^{bpq}i^*(\Omega^{-1}C_{mnpq})$

$$*\underline{K}^{ab}=\underline{\varepsilon}^{amn}\underline{\varepsilon}^{bpq}i^*(\tfrac{1}{2}\Omega^{-1}\varepsilon^{rs}_{\ \ mn}C_{rspq})=i^*(\Omega^{-1}C^{ambn}n_m n_n)$$

$$N_{ab}=\underline{g}_{ac}\,i^*(R^c_b-\tfrac{1}{6}R\delta^c_b)-\mathscr{R}\underline{g}_{ab}$$

(where $\mathscr{R}$ is the scalar curvature of the 2-sphere base space of $\mathscr{I}$), which are the electric and magnetic components of the spin 2-field and the News tensor at $\mathscr{I}$, respectively.

## 3. CONFORMAL ISOMETRY AND CONSTANT EXPANSION

We first prove that the expansion associated with a conformal Killing vector field must be constant.

*Theorem 1.* Let $(\tilde{M}, \tilde{g}_{ab})$ be a space-time that is (i) vacuum, (ii) asymptotically NUT (Magnon, 1987a,b; Magnon and Guo, 1987), (iii) with nonvanishing mass. Let $\xi^a$ s.t. $\mathscr{L}_\xi \tilde{g}_{ab} = k\tilde{g}_{ab}$. Then $k$ must be constant (not necessarily vanishing).

*Proof.* Let $\tilde{k}_a = \tilde{\nabla}_a k$. This is one of the conformal Killing data of $\xi$, and satisfies

$$\tilde{\nabla}_a \tilde{k}_b = -\xi^c \tilde{\nabla}_c \tilde{S}_{ab} - k\tilde{S}_{ab} - 2\tilde{g}^{cd}\tilde{S}_{c(a}\tilde{\nabla}_{b)}\tilde{\xi}_d \tag{2}$$

where $\tilde{S}_{ab} = \tilde{R}_{ab} - \frac{1}{6}\tilde{R}\tilde{g}_{ab}$.

Since the space-time is vacuum, $\tilde{\nabla}_a \tilde{k}_b = 0$: $\tilde{k}^a$ is a constant Killing vector field. Furthermore, the affine colineation equation

$$\tilde{\nabla}_a \tilde{\nabla}_b \tilde{k}_c = C_{abcd}\tilde{k}^d = 0 \tag{3}$$

implies that $\tilde{k}^c$ must be a constant null field $\alpha n^c$ in the neighborhood of any point where $C_{abcd} \neq 0$, thus inducing a supertranslation $\underline{\alpha}\underline{n}^c$ at $\mathscr{I}$. The action of the mass (resp. dual mass) on a supertranslation has been introduced as follows (Magnon, 1986a).

Let $l_a$ be a covector on $\mathscr{I}$ such that $\underline{n}^a l_a = -1$. If a gravitational magnetic monopole is present, the assumption $N_{ab} = 0$ is reasonable, implying the following definitions of the total mass $M$ and magnetic mass $M^*$, respectively:

$$M = \int_{\mathscr{C}} \underline{\alpha}\underline{\varepsilon}_{abc}\, {}^*\underline{K}^{cm}\underline{l}_m \, dS^{ab} \tag{4}$$

$$M^* = \int_{\mathscr{C}} \underline{\alpha}\underline{\varepsilon}_{abc}\underline{K}^{cm}\underline{l}_m \, dS^{ab} \tag{5}$$

for any cross section $\mathscr{C}$ of $\mathscr{I}$, and any supertranslation $\alpha\underline{n}^a$.

Since

$$\begin{aligned} \underline{\alpha}\, {}^*\underline{K}^{ab} &= i^*(\Omega^{-1}C^{ambn}\alpha n_m n_n) \\ &= \underline{K}^{ambn}\underline{\alpha}\underline{n}_m\underline{n}_n \end{aligned}$$

then $C_{abcd}\alpha n^d = 0$ (for the supertranslation induced by $\xi$) implies that $\alpha$ must vanish if the total mass $M$ does not vanish. This implies that $k$ must be constant in a suitable neighborhood of $\mathscr{I}$.

We now proceed to show that $k$ does not vanish in the generic case: $\xi^a$ is a conformal isometry generating a constant dilation.

Recall that in the absence of radiation ($N_{ab} = 0$)

$$M = \int_{\mathscr{S}} \underset{\sim}{\alpha} f \, d\mathscr{S}, \qquad M^* = \int_{\mathscr{S}} \underset{\sim}{\alpha} g \, d\mathscr{S}$$

where $f$ and $g$ are functions on $\mathscr{S}$ (Magnon, 1986a).

Furthermore, it is easy to show that

$$\mathscr{L}_\xi C_{abc}{}^d = 0, \qquad \mathscr{L}_\xi g_{ab} = g_{ab}(k + 2\Omega^{-1}\mathscr{L}_\xi \Omega) = \chi g_{ab}$$

imply

$$\mathscr{L}_{\underset{\sim}{\xi}} \underset{\sim}{g}_{ab} = \chi \underset{\sim}{g}_{ab}, \qquad \text{where } \underset{\sim}{\xi} = i^*(\xi) \tag{6}$$

$$\mathscr{L}_{\underset{\sim}{\xi}} n^a = \tfrac{1}{2}(k - \chi)\underset{\sim}{n}^a \tag{7}$$

$$\mathscr{L}_{\underset{\sim}{\xi}} \underset{\sim}{K}^{ab} = \tfrac{1}{2}(k - 5\chi)\underset{\sim}{K}^{ab} \tag{8}$$

$$\mathscr{L}_{\underset{\sim}{\xi}} {}^*\underset{\sim}{K}^{ab} = \tfrac{1}{2}(k - 5\chi)\, {}^*\underset{\sim}{K}^{ab} \tag{9}$$

$$\mathscr{L}_{\underset{\sim}{\xi}} f = -\tfrac{1}{2}(k + 3\chi) f, \qquad \mathscr{L}_{\underset{\sim}{\xi}} g = -\tfrac{1}{2}(k + 3\chi) g \tag{10}$$

Let $u$ denote an affine parameter along the orbits of $\underset{\sim}{\xi}$. Then

$$df/f = -\tfrac{1}{2}(k + 3\chi)\, du$$

implies

$$f(u) = f(0) \exp\left[ -\frac{1}{2} ku - \frac{3}{2} \int_0^u \chi(\tilde{u})\, d\tilde{u} \right] \tag{11}$$

Thus, $f(0)$ and $g(0)$ govern the values of $\underset{\sim}{K}^{ab}$ and ${}^*\underset{\sim}{K}^{ab}$—the characteristic datum at $\mathscr{I}$ (Magnon, 1986a).

Since $\mathscr{I}$ is compact, if the orbits of $\underset{\sim}{\xi}$ are closed, a reasonable assumption (Magnon, 1987a), $u$ and $\int_0^u \chi(\tilde{u})\, d\tilde{u}$ take bounded values, thus enabling nonzero values of $k$: $\xi$ is a conformal isometry generating a constant dilation (The result also could have been derived from the expression for g.) ■

## 4. MAGNETIC MASS (OR CHARGE) TO MASS (OR ELECTRIC CHARGE) RATIO

We know from Magnon (1986a) that, in the absence of radiation ($N_{ab} = 0$), an infinitesimal supertranslation $\alpha n^a$ ($\alpha$ a function on $\mathscr{S}$) can be restricted, provided $\alpha$ is everywhere positive, to $\alpha = \text{const}$.

This follows from the gauge-invariant equation

$$\underset{=}{g}^{mn} D_m \alpha D_n \alpha - \underset{=}{g}^{mn} D_m D_n \alpha - \tfrac{1}{2} \mathcal{R} \alpha^2 = \text{const} \tag{12}$$

This in turn implies that $M$ and $M^*$ are characterized by the values of $f$ and $g$, respectively.

The explicit expressions (11) for $f(u)$ and $g(u)$ at any point on an orbit of $\underset{=}{\xi}$ imply the following:

*Theorem.* Given an asymptotically NUT space-time, the value of the ratio of the total magnetic mass over the total mass is given by $f(0)/g(0)$, and is thus provided by the characteristic datum of the space time at $\mathcal{I}$.

A similar derivation can be accomplished if a (source-free) Maxwell field is available on the space-time under consideraion. Since the vacuum condition ($\mathcal{R}_{ab} = 0$) is no longer available in this case, one has to assume, in addition to the asymptotic NUT-like structure at $\mathcal{I}$, the existence of a conformal isometry inducing a constant dilation.

For a Maxwell field $F_{ab}$ s.t. $\mathcal{L}_\xi F_{ab} = 0$, conditions similar to conditions (8)-(10) can be derived for $E^a$ and $B^a$, the electric and magnetic components of $F_{ab}$ with respect to $\xi$ (a null observer). The values $f(0)$ and $g(0)$ can be chosen as characteristic data for $F_{ab}$, thus implying an identification of the two ratios.

One can summarize the situation as follows. Each gravitational magnetic monopole solution to Einstein's equation with conformal Killing vector field associated with a constant dilation selects a specific Maxwellian magnetic monopole solution such that the ratios of the magnetic and electric charges can be identified. This can be interpreted as a unifying feature between gravity and electromagnetism via solutions that could be useful in the early universe and cosmology.

## REFERENCES

Magnon, A. (1986a). *Journal of Mathematical Physics*, **27**, 1059.

Magnon, A. (1986b). *Journal of Mathematical Physics*, **27**, 1066.

Magnon, A. (1986c). *Journal of Mathematical Physics*, **27**, 2105.

Magnon, A. (1987a). Mass, dual mass and gravitational entropy, *Journal of Mathematical Physics*, to appear.

Magnon, A. (1987b). Unification of gravity and electromagnetism in dimension 4: Some peculiar aspects, Preprint.

Magnon, A., and Guo, H. Y. (1987). Mass and dual mass, a gravitational analogue of the Dirac quantization rule and its physical implications, *General Relativity and Gravitation*, to appear.